

On orbits of characters of reductive groups over CDVR

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Groups over a local ring

K - a non-archimedean local field

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 $\mathcal{O}, \pi$ - its ring of integers, with uniformiser π $\mathbb{F}_q$ - its residue field

On orbits of characters of reductive p -pairs of

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Our focus:

$$\mathrm{SmoothRep}(\mathbb{G}(\mathcal{O})) \quad \longleftrightarrow \quad \bigcup_{r \in \mathbb{Z}_{\geq 0}} \mathrm{Rep}(\mathbb{G}(\mathcal{O}/\pi^r))$$

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☒ On orbits of characters of reductive groups

Groups over a local ring

Let us be more rigorous.

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The $r = 1$ case: repn theory has been extensively studied

- GL_2, SL_2 : Jordan, Schur (1900s)
- GL_3 : Steinberg(1951)
- GL_n : Green(1955), Lusztig–Srinivasan(1977), Zelevinsky(1981), Jing–Wu(2023) ...
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This talk: The $r > 1$ case

In this case the representations are far from been well-understood

On orbits of characters of reductive groups over CDVR

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2. Fix an (arbitrary) non-trivial character $\psi: \mathbb{F}_q \rightarrow \mathbb{C}^\times$.

Orbits ($r > 1$)

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via $\mu(-, -)$ and ψ

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Definition

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SLOGAN: Orbits in $\mathfrak{g}$ reveal the nature of $\text{Rep}(\mathbb{G}(\mathcal{O}/\pi^r))$.

We illustrate this slogan with two cases: SL_2 and GL_n .

The 1st case: SL_2 (via class numbers and modular forms)

A natural question: When is $\mathcal{O}_F$ a UFD?
(F -number field, $\mathcal{O}_F$ -ring of integers in F .)

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Proposition

$\mathcal{O}_F$ is a UFD iff the class number of F is 1.

F is an $\mathcal{O}_F$ -module
(not finitely generated)

$J_F := \{\text{finitely generated submodules of } F\}$

$P_F := \{\text{submodules generated by one elements}\}$

Proposition (Definition of class number)

*The natural group J_F/P_F formed by multiplication is finite;
its **cardinal** is called the **class number** of F .*

Examples:

The class numbers of $\mathbb{Q}$ and of $\mathbb{Q}[i]$ are 1.

The class number of $\mathbb{Q}[\sqrt{-5}]$ is 2.

$$N \in \mathbb{Z}_{>0}$$

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$S_2(N) :=$ space of cusp forms of weight 2 and level $\Gamma(N)$:

- certain holomorphic functions on the upper-half plane
- a finite dimensional space over $\mathbb{C}$
- $\mathrm{SL}_2(\mathbb{Z}/N)$ acts on $S_2(N)$ by

$$\gamma: f(z) \mapsto (cz + d)^{-2} f(\gamma^{-1}(z))$$

for $\gamma^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; here $\gamma^{-1}(z)$ is the Möbius transformation.

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In particular: $S_2(N) := \bigoplus_{\sigma \in \mathrm{Irr}(\mathrm{SL}_2(\mathbb{Z}/N))} m_{\sigma} \cdot \sigma$.

Recall the irr characters of $\mathrm{SL}_2(\mathbb{F}_p)$:
 (p is an odd prime)

| | | | | | | |
|-----|---|-----|-----------------|-----------------|-----------------|-----------------|
| dim | 1 | p | $p+1$ | $\frac{p+1}{2}$ | $p-1$ | $\frac{p-1}{2}$ |
| # | 1 | 1 | $\frac{p-3}{2}$ | 2 | $\frac{p-1}{2}$ | 2 |

$\implies \exists$ two irreps of $\text{dim} = \frac{p-1}{2}$; denote them by σ_+ and σ_- .

(sometimes called the **degenerate cuspidal representations**)

Theorem (Hecke, 1920s)

Suppose that $p > 3$ and $p \equiv 3 \pmod{4}$, and let $N = p$. Then

$$m_{\sigma_+} - m_{\sigma_-} = h(-p),$$

where $h(-p)$ denotes the class number of $\mathbb{Q}(\sqrt{-p})$.

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- $\mathrm{Sp}_4(\mathbb{F}_p)$ (using Siegel modular forms): Lee–Weintraub (1982, 1989), Tsushima (1984), Hashimoto (1984), ...

Using orbits, Hecke's result can be extended to $\mathrm{SL}_2(\mathbb{Z}/p^r)$:

Theorem (C.–Feng, 2025)

Suppose that $p > 3$ and $p \equiv 3 \pmod{4}$, and $r > 1$. Let $N = p^r$. Then

$$\sum_{\sigma' \in \mathcal{U}_+} m_{\sigma'} - \sum_{\sigma'' \in \mathcal{U}_-} m_{\sigma''} = p^{r-1} \cdot h(-p),$$

where $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) denotes the elements in $\mathrm{Irr}(\mathrm{SL}_2(\mathbb{Z}/p^r))$ whose orbit contains $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (resp. $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$).

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$$\sum_{\sigma' \in \mathcal{U}_+} m_{\sigma'} - \sum_{\sigma'' \in \mathcal{U}_-} m_{\sigma''} = p^{r-1} \cdot h(-p),$$

where $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) denotes the elements in $\mathrm{Irr}(\mathrm{SL}_2(\mathbb{Z}/p^r))$ whose orbit contains $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (resp. $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$).

$\implies$ This result suggests that **regular nilpotent orbit** irreps are higher level analogues of degenerate cuspidal irreps.

Any conceptual explanation?

The 2nd case: GL_n
(via higher Deligne–Lusztig theory)

Higher level Deligne–Lusztig theory:

A geometric way to construct representations of $\mathbb{G}(\mathcal{O}/\pi^r)$

They are of the form $R_{T_r, U_r}^\theta := \sum_i (-1)^i H_c^i(S_{T_r, U_r}, \overline{\mathbb{Q}}_\ell)_\theta$

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- $r > 1$: Stasinski (2009) - completed the proofs for general $\mathcal{O}$

From now on: $\mathbb{G} = \mathrm{GL}_n$.

The character formula in the case of GL_n :

Theorem (C.–Stasinski(2017 for even r , 2023 for odd r))

Suppose that θ is regular(= “r.s.s. in $\mathfrak{gl}_n$ ”). Then

$$R_{T_r, U_r}^\theta \cong ((-1)^{\epsilon_{G_1} + \epsilon_{T_1}})^r \cdot \text{Ind}_{(T_r G_r')^F}^{G_r^F} \hat{\rho}_\theta,$$

where $\hat{\rho}_\theta \in \text{Irr}((T_r G_r')^F)$ is uniquely characterised by:

(a) *If $s \in (T_r^1 G_r')^F$, then*

$$\text{Tr}(s, \hat{\rho}_\theta) = \left(\frac{1}{q^{n(n-1)/2}} \right)^{l-l'} \cdot \text{Tr} \left(s, \text{Ind}_{(T_r^1 G_r')^F}^{(T_r^1 G_r')^F} \tilde{\theta}|_{(T_r^1 G_r')^F} \right);$$

(b) *if $s \in T_1^F$, then $\text{Tr}(s, \hat{\rho}_\theta) = \text{St}_{G_1}(s)^{l-l'} \cdot \theta(s)$.*

The evaluation at $1 \in GL_n(\mathcal{O}/\pi^r)$ gives a simple dim formula:

Corollary (C.–Stasinski(2017 for even r , 2023 for odd r))

Suppose that θ is regular. Then:

$$\dim R_{T_r, U_r}^\theta = (-1)^{r \cdot (\mathrm{rk}_q T_1 + n)} \cdot \frac{|G_1^F|_{p'}}{|T_1^F|} \cdot q^{\frac{n(n-1)}{2} \cdot (r-1)}$$

The dimension formula works equally well **for $r = 1$** ;

the *character formula* extends to general $\mathbb{G}$, at the cost of:

- “ $q \geq 7$ ” (no requirement on $p := \mathrm{char}(\mathbb{F}_q)$);
- strongly generic ($\approx$ “strongly reg.” introduced by Steinberg)

As a by-product:

work of Hill(1995) + properties of higher DL reps

$$\implies (\text{for } r > 1)$$

a characterisation of reg s.s. orbits in $\mathfrak{gl}_n$:

Theorem (C.–Stasinski (2023))

The regular semisimple orbit irreducible reps of $\mathrm{GL}_n(\mathcal{O}/\pi^r)$

are precisely

the regular higher level Deligne–Lusztig reps up to ± 1 .

- The sign ± 1 is determined by the dimension formula.

Example: $\text{Irr}(\text{GL}_2(\mathbb{F}_q[[\pi]]/\pi^2))$

Irreps inflated from $\text{GL}_2(\mathbb{F}_q)$ (possibly tensored by 1-dim irreps):

| | | | | |
|-----|-----------|-----------|-----------------|--------------|
| dim | 1 | q | $q+1$ | $q-1$ |
| # | $q^2 - q$ | $q^2 - q$ | $q(q-1)(q-2)/2$ | $q^2(q-1)/2$ |

Irreps *NOT* come from inflations:

| | | | |
|-----|--------------|-------------------|-------------|
| dim | $q^2 + q$ | $q^2 - q$ | $q^2 - 1$ |
| # | $q(q-1)^3/2$ | $q(q+1)(q-1)^2/2$ | $q^3 - q^2$ |

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red ones: reg. s.s. orbit irreps=reg. higher DL reps (signs= +1).

blue ones: appear in irreg. higher DL reps

brown ones: single Jordan block irreps; some missed by higher DL

(All of them appear in (both) Stasinski (2011) and C.(2020).)

Finally, some open questions.

Open problems

$\dim$ of R_{T_r, U_r}^θ without any conditions on the parameters?

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Conjecture (C. 2024)

For general $\mathbb{G}$, r , and DL rep R_{T_r, U_r}^θ :

$$\operatorname{sgn}(R_{T_r, U_r}^\theta) = \operatorname{sgn}(R_{T_1, U_1}^1) \left(1 + \frac{\log_q |\dim R_{T_r, U_r}^\theta|_p}{\#\Phi^+} \right)$$

where $(-)_p$ denotes the p -part of a positive integer.

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Known cases:

- T_r splits or $\mathbb{G}$ is a torus (view RHS as 1, as $\operatorname{sgn}(R_{T_1, U_1}^\theta) = 1$);
- $r = 1$ (Deligne–Lusztig, 1976);
- $\mathbb{G} = \operatorname{GL}_n$ or SL_n , and θ is regular (C.–Stasinski, 2017, 2023);
- $q \geq 7$, and θ is strongly generic (C.–Stasinski, 2017, 2023);
- T_1 is elliptic and $\theta = 1$ (C., 2024 ($q \geq 7$), and C. Chan 2024);
- $\mathbb{G} = \operatorname{GL}_2$ or SL_2 (C., 2024).

Definition

Let C be the set of conjugacy classes of $\mathbb{G}(\mathcal{O}/\pi^r)$. Then

$$n_F : [g^{-1}F(g)] \longmapsto [F(g)g^{-1}]$$

is a self-bijection on C , called the twisting operator.

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Expectation (C. 2022):

Let $\mathbb{G} = \text{SL}_n$. Then for primitive ρ , $\Omega(\rho)$ is **s.s.** iff $\chi_\rho \circ n_F = \chi_\rho$.

- This is true for $\text{SL}_2(\mathbb{F}_q[[\pi]]/\pi^2)$ with odd q .
- The “only if” part holds when $\Omega(\rho)$ is regular and semisimple.

Specialise the two open questions to primitive irreps of $\mathrm{SL}_2(\mathbb{F}_q[[\pi]]/\pi^2)$

Example: Primitive irreps of $\mathrm{SL}_2(\mathbb{F}_q[[\pi]]/\pi^2)$ (q odd)

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|-----|---------------------|-------------------|-------------------|
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| # | $\frac{(q-1)^2}{2}$ | $\frac{q^2-1}{2}$ | $4q$ |

red ones: reg. s.s. orbit irreps.

blue ones: reg. nilp. orbit irreps.

Two higher level D–L varieties: [Ref: Lusztig(2004)]

- X_1 : a disjoint union of affine planes $/\overline{\mathbb{F}}_q$
- X_2 : an irreducible algebraic surface $/\overline{\mathbb{F}}_q$

red ones: in either $H_c^4(X_1)$ or $H_c^2(X_2)$ (so signs are all $+1$).

blue ones: some in $H_c^4(X_1)$; others are not in any $H_c^i(X_j)$.

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red ones: n_F is the identity map.

blue ones: n_F is an involution without fixed points.

Thank you!