On orbits of characters of reductive groups over CDVR

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This talk: only char$=0$ representations.

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Example

If $r = 2$ and $\mathcal{O} = \mathbb{F}_q[[\pi]]$, then $\mathbb{G}(\mathcal{O}/\pi^2) \cong \mathbb{G}(\mathbb{F}_q) \ltimes \mathfrak{g}(\mathbb{F}_q)$

The $r = 1$ case: repn theory has been extensively studied

- $\mathrm{GL}_2, \mathrm{SL}_2$: Jordan, Schur (1900s)
- GL_3: Steinberg(1951)
- GL_n: Green(1955), Lusztig–Srinivasan(1977), Zelevinsky(1981), Jing–Wu(2023) ...
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This talk: The $r > 1$ case

In this case the representations are far from been well-understood

Orbits ($r > 1$)

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$$\mu \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{A}^1.$$

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2. Fix an (arbitrary) non-trivial character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$.

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We illustrate this slogan with two cases: SL_2 and GL_n.

The 1st case: SL_2
(via class numbers and modular forms)

An observation: $\mathbb{Z}$ is a UFD.

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... and sometimes fails: In $\mathbb{Z}[\sqrt{-5}]$: $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.

A natural question: When is $\mathcal{O}_F$ a UFD?
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Proposition

$\mathcal{O}_F$ is a UFD iff the class number of F is 1.

F is an $\mathcal{O}_F$-module
(not finitely generated)

$J_F := \{\text{finitely generated submodules of } F\}$

$P_F := \{\text{submodules generated by one elements}\}$

Proposition (Definition of class number)

The natural group J_F/P_F formed by multiplication is finite; its cardinal is called the class number of F.

Examples:

The class numbers of $\mathbb{Q}$ and of $\mathbb{Q}[i]$ are 1.

The class number of $\mathbb{Q}[\sqrt{-5}]$ is 2.

Class numbers and modular forms

$$N \in \mathbb{Z}_{>0}$$

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$S_2(N) :=$ space of cusp forms of weight 2 and level $\Gamma(N)$:

- certain holomorphic functions on the upper-half plane
- a finite dimensional space over $\mathbb{C}$
- $\mathrm{SL}_2(\mathbb{Z}/N)$ acts on $S_2(N)$ by

$$\gamma\colon f(z) \longmapsto (cz+d)^{-2} f(\gamma^{-1}(z))$$

for $\gamma^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; here $\gamma^{-1}(z)$ is the Möbius transformation.

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In particular: $S_2(N) := \bigoplus_{\sigma \in \mathrm{Irr}(\mathrm{SL}_2(\mathbb{Z}/N))} m_\sigma \cdot \sigma$.

Recall the irr characters of $\mathrm{SL}_2(\mathbb{F}_p)$:
(p is an odd prime)

dim	1	p	$p+1$	$\frac{p+1}{2}$	$p-1$	$\frac{p-1}{2}$
#	1	1	$\frac{p-3}{2}$	2	$\frac{p-1}{2}$	2

$\implies \exists$ two irreps of dim$=\frac{p-1}{2}$; denote them by σ_+ and σ_-.

(sometimes called the degenerate cuspidal representations)

Theorem (Hecke, 1920s)

Suppose that $p > 3$ and $p \equiv 3 \mod 4$, and let $N = p$. Then

$$m_{\sigma_+} - m_{\sigma_-} = h(-p),$$

where $h(-p)$ denotes the class number of $\mathbb{Q}(\sqrt{-p})$.

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- $\mathrm{Sp}_4(\mathbb{F}_p)$ (using Siegel modular forms): Lee–Weintraub (1982, 1989), Tsushima (1984), Hashimoto (1984), ...

Using orbits, Hecke's result can be extended to $\mathrm{SL}_2(\mathbb{Z}/p^r)$:

Theorem (C.–Feng, 2025)

Suppose that $p > 3$ and $p \equiv 3 \mod 4$, and $r > 1$. Let $N = p^r$. Then

$$\sum_{\sigma' \in \mathcal{U}_+} m_{\sigma'} - \sum_{\sigma'' \in \mathcal{U}_-} m_{\sigma''} = p^{r-1} \cdot h(-p),$$

where $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) denotes the elements in $\mathrm{Irr}(\mathrm{SL}_2(\mathbb{Z}/p^r))$ whose orbit contains $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\left(resp. \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \right)$.

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$\Longrightarrow$ This result suggests that regular nilpotent orbit irreps are higher level analogues of degenerate cuspidal irreps.

Any conceptual explanation?

The 2nd case: GL_n

(via higher Deligne–Lusztig theory)

Higher level Deligne–Lusztig theory:

A geometric way to construct representations of $\mathbb{G}(\mathcal{O}/\pi^r)$

They are of the form $R^{\theta}_{T_r,U_r} := \sum_i (-1)^i H^i_c(S_{T_r,U_r}, \overline{\mathbb{Q}}_\ell)_\theta$

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- $r > 1$: Stasinski (2009) - completed the proofs for general $\mathcal{O}$

$r = 1$: Deligne–Lusztig representations are VERY NICE:

(1) every irrep appears in some $R^{\theta}_{T_1,U_1}$

(2) most $R^{\theta}_{T_1,U_1}$ are irreducible (up to ± 1)

(3) a nice formula for r.s.s. elements
(very difficult for general elements)

(4) a beautiful dimension formula (including the sign formula)

$r > 1$: Deligne–Lusztig representations are ~~*VERY*~~ NICE:

(1) (exhaustion) fails; one shall generalise higher DL: Stasinski(2011) and C.(2020)
(two different methods; not well-understood yet)

(2) (irr) extends generically: Lusztig(2004), Stasinski(2009)

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(3 & 4 ++) a character formula for R_{T_r,U_r}^{θ} for $\forall g \in \mathbb{G}(\mathcal{O}/\pi^r)$:
C.–Stasinski (2017,2023)

- not available for $r = 1$;
- no requirement on $p := \mathrm{char}(\mathbb{F}_q)$.

From now on: $\mathbb{G} = \mathrm{GL}_n$.

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Let $l = [\frac{r+1}{2}],\ l' = [\frac{r}{2}]$.

The character formula in the case of GL_n:

Theorem (C.–Stasinski(2017 for even r, 2023 for odd r))

Suppose that θ is regular(="r.s.s. in $\mathfrak{gl}_n$"). Then

$$R_{T_r,U_r}^{\theta} \cong ((-1)^{\epsilon_{G_1}+\epsilon_{T_1}})^r \cdot \mathrm{Ind}_{(T_rG_r^{l'})^F}^{G_r^F} \hat{\rho}_\theta,$$

where $\hat{\rho}_\theta \in \mathrm{Irr}((T_rG_r^{l'})^F)$ is uniquely characterised by:

(a) *If $s \in (T_r^1G_r^{l'})^F$, then*

$$\mathrm{Tr}(s,\hat{\rho}_\theta) = \left(\frac{1}{q^{n(n-1)/2}}\right)^{l-l'} \cdot \mathrm{Tr}\left(s, \mathrm{Ind}_{(T_r^1G_r^{l})^F}^{(T_r^1G_r^{l'})^F} \widetilde{\theta}|_{(T_r^1G_r^{l})^F}\right);$$

(b) *if $s \in T_1^F$, then $\mathrm{Tr}(s,\hat{\rho}_\theta) = \mathrm{St}_{G_1}(s)^{l-l'} \cdot \theta(s)$.*

The evaluation at $1 \in \mathrm{GL}_n(\mathcal{O}/\pi^r)$ gives a simple dim formula:

Corollary (C.–Stasinski(2017 for even r, 2023 for odd r))

Suppose that θ is regular. Then:

$$\dim R_{T_r,U_r}^{\theta} = (-1)^{r\cdot(\mathrm{rk}_q T_1 + n)} \cdot \frac{|G_1^F|_{p'}}{|T_1^F|} \cdot q^{\frac{n(n-1)}{2}\cdot(r-1)}$$

The dimension formula works equally well for $r = 1$;

the *character formula* extends to general $\mathbb{G}$, at the cost of:

- "$q \geq 7$" (no requirement on $p := \mathrm{char}(\mathbb{F}_q)$);
- strongly generic($\approx$"strongly reg." introduced by Steinberg)

As a by-product:
work of Hill(1995) + properties of higher DL reps
$\Longrightarrow$ (for $r > 1$)
a characterisation of reg s.s. orbits in $\mathfrak{gl}_n$:

Theorem (C.–Stasinski (2023))

The regular semisimple orbit irreducible reps of $\mathrm{GL}_n(\mathcal{O}/\pi^r)$

are precisely

the regular higher level Deligne–Lusztig reps up to ± 1.

- The sign ± 1 is determined by the dimension formula.

Example: $\mathrm{Irr}(\mathrm{GL}_2(\mathbb{F}_q[[\pi]]/\pi^2))$

Irreps inflated from $\mathrm{GL}_2(\mathbb{F}_q)$ (possibly tensored by 1-dim irreps):

dim	1	q	$q+1$	$q-1$
#	q^2-q	q^2-q	$q(q-1)(q-2)/2$	$q^2(q-1)/2$

Irreps *NOT* come from inflations:

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red ones: reg. s.s. orbit irreps=reg. higher DL reps (signs$=+1$).
blue ones: appear in irreg. higher DL reps
brown ones: single Jordan block irreps; some missed by higher DL

(All of them appear in (both) Stasinski (2011) and C.(2020).)

Finally, some open questions.

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Conjecture (C. 2024)

For general $\mathbb{G}$, r, and DL rep $R^\theta_{T_r,U_r}$:

$$\operatorname{sgn}(R^\theta_{T_r,U_r}) = \operatorname{sgn}(R^1_{T_1,U_1})^{\left(1+\frac{\log_q |\dim R^\theta_{T_r,U_r}|_p}{\#\Phi^+}\right)}$$

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Known cases:

- T_r splits or $\mathbb{G}$ is a torus (view RHS as 1, as $\mathrm{sgn}(R^{\theta}_{T_1,U_1})=1$);
- $r=1$ (Deligne–Lusztig,1976);
- $\mathbb{G}=\mathrm{GL}_n$ or SL_n, and θ is regular (C.–Stasinski,2017,2023);
- $q\geq 7$, and θ is strongly generic (C.–Stasinski,2017,2023);
- T_1 is elliptic and $\theta=1$ (C.,2024 ($q\geq 7$), and C.Chan 2024);
- $\mathbb{G}=\mathrm{GL}_2$ or SL_2 (C.,2024).

Definition

Let C be the set of conjugacy classes of $\mathbb{G}(\mathcal{O}/\pi^r)$. Then

$$n_F\colon [g^{-1}F(g)] \longmapsto [F(g)g^{-1}]$$

is a self-bijection on C, called the twisting operator.

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Question: relations between n_F and orbits of $\mathrm{Irr}(\mathbb{G}(\mathcal{O}/\pi^r))$?
(for $r > 1$)

Definition

Let C be the set of conjugacy classes of $\mathbb{G}(\mathcal{O}/\pi^r)$. Then

$$n_F \colon [g^{-1}F(g)] \longmapsto [F(g)g^{-1}]$$

is a self-bijection on C, called the twisting operator.

Question: relations between n_F and orbits of $\mathrm{Irr}(\mathbb{G}(\mathcal{O}/\pi^r))$?
(for $r > 1$)

Expectation (C. 2022):
Let $\mathbb{G} = \mathrm{SL}_n$. Then for primitive ρ, $\Omega(\rho)$ is s.s. iff $\chi_\rho \circ n_F = \chi_\rho$.

- This is true for $\mathrm{SL}_2(\mathbb{F}_q[[\pi]]/\pi^2)$ with odd q.
- The "only if" part holds when $\Omega(\rho)$ is regular and semisimple.

Example: Primitive irreps of $\mathrm{SL}_2(\mathbb{F}_q[[\pi]]/\pi^2)$ (q odd)

dim	q^2+q	q^2-q	$\frac{q^2-1}{2}$
$\#$	$\frac{(q-1)^2}{2}$	$\frac{q^2-1}{2}$	$4q$

red ones: reg. s.s. orbit irreps.

blue ones: reg. nilp. orbit irreps.

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Two higher level D–L varieties: [Ref: Lusztig(2004)]

- X_1 : a disjoint union of affine planes $/\overline{\mathbb{F}}_q$
- X_2 : an irreducible algebraic surface $/\overline{\mathbb{F}}_q$

red ones: in either $H_c^4(X_1)$ or $H_c^2(X_2)$ (so signs are all $+1$).
blue ones: some in $H_c^4(X_1)$; others are not in any $H_c^i(X_j)$.

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red ones: n_F is the identity map.
blue ones: n_F is an involution without fixed points.

Thank you!